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Growth Optimization with Downside Protection: A New Paradigm for Portfolio Selection

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This study introduces growth optimization with downside protection as a portfolio selection technique based on power-log utility functions that combine long-term portfolio growth maximization with the behavioral tenets of prospect theory. This simple but powerful technique can be used with all types of assets, including those with highly skewed and fat-tailed return distributions. We use three assets with very different types of return distributions to show how effective this technique is in constructing portfolios with positively skewed returns that combine high upside potential with downside protection.

Portfolio selection is the process of selecting assets and their relative weights to construct the best investment portfolio for a given investor. Harry Markowitz [1952a] defined the mean-variance framework for portfolio construction, and it is still the most popular method. His technique selects a portfolio that minimizes risk for a given level of reward, where reward is the mean of the portfolio return distribution and risk is measured as the variance of the portfolio return distribution.

Another approach to portfolio selection is based on multiperiod portfolio theory. To maximize long-term portfolio growth over time, portfolios are selected using the log utility function and maximizing expected utility as defined by Von Neumann and Morgenstern [1944] and Savage [1964]. A related portfolio selection method models investor preferences with the family of power utility functions. Both the log and power utility functions have the myopic property, which allows us to solve multiperiod portfolio selection problems by converting them to one-period problems.

Portfolio selection methods based on log and power utility functions have been discussed extensively in the literature. For examples, see Kelly [1956], Markowitz [1959, 1976], Latane [1959], Brieman [1960, 1961], Pratt [1964], Latane and Tuttle [1967], Mossin [1968], Samuelson [1969, 1971, 1979], Merton [1971], Hakansson [1971, 1974, 1979], Leland [1972], Ross [1974], Merton and Samuelson [1974], Huberman and Ross [1980], Grauer and Hakansson [1982, 1985, 1993], and MacLean and Ziemba [1999].

Behavioral finance has given us additional insight into investor actions and preferences. Barberis and Thaler's [2002] extensive survey of behavioral finance includes studies about investor psychology. They point to prospect theory as "the most successful at capturing the experimental results." Prospect theory is a descriptive theory proposed by Kahneman and Tversky [1979], who defined utility over gains and losses, an idea also proposed by Markowitz [1952b]. Prospect theory aims to explain the choices made by people when offered a gamble with two outcomes, one a gain and the other a loss.

Tversky and Kahneman [1991, 1992] extended their analysis of choice under uncertainty to riskless choice and to gambles with more than two outcomes. They define the S-shaped utility function shown in Figure 1, and describe it as follows:

> Outcomes of risky prospects are evaluated by a value function that has three essential characteristics. Reference dependence: the carriers of value are gains and losses defined relative to a reference point. Loss aversion: the function is steeper in the negative than in the positive domain; losses loom larger than corresponding gains. Diminishing sensitivity: the marginal value of both gains and losses decreases with their size. These properties give rise to an asymmetric S-shaped value function, concave above the reference point and convex below it (Tversky and Kahneman [1991, p. 1039]).

In this study, we combine the multiperiod portfolio selection techniques that use the log and power utility functions with prospect theory's ideas of reference dependence, loss aversion, and diminishing sensitivity to gains. We develop the growth optimization with downside protection portfolio selection technique. It is a simple but powerful technique that can be used with all types of assets, including those with highly skewed and fat-tailed return distributions. We use three assets with very different types of return distributions to show our technique's effectiveness in constructing portfolios

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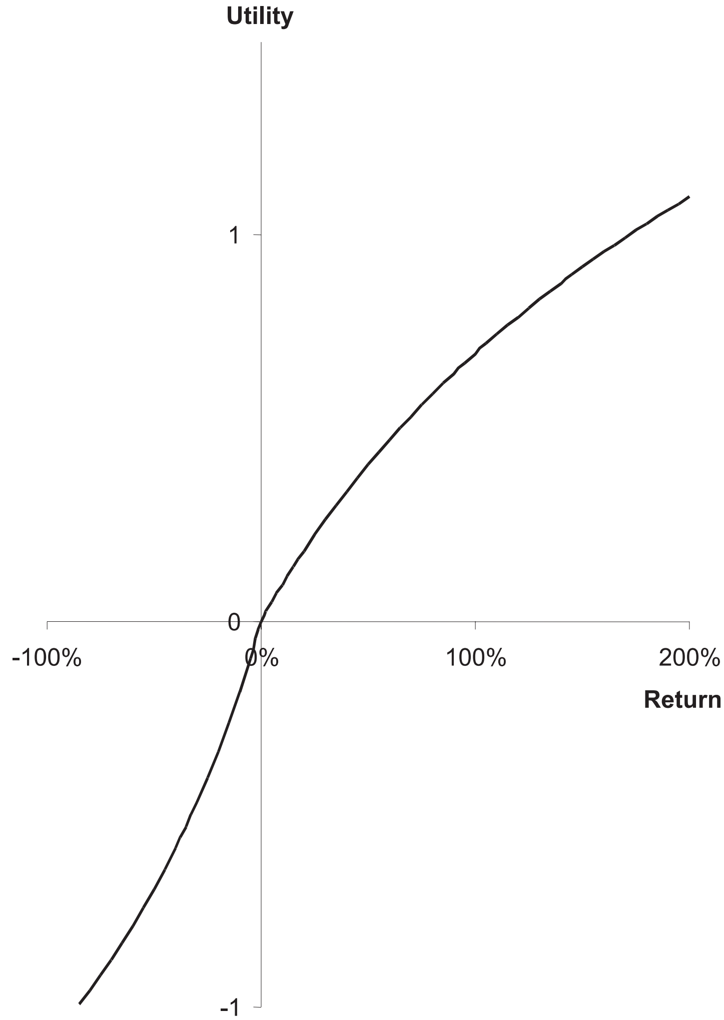
FIGURE 1
Prospect Theory's S-Shaped Utility Function



with positively skewed returns that combine high upside potential with downside protection.

The next section describes our methodology for portfolio selection using growth optimization with downside protection. We then discuss the optimal portfolios constructed with this technique, comparing their compositions with those of portfolios produced using classic Markowitz mean-variance optimization and standard power utility functions. The last section gives our conclusions.

Methodology

Maximizing long-term portfolio growth is the goal of all investors, whether they are building retirement wealth or endowment wealth. If an investor wants to maximize portfolio growth without restriction, the appropriate utility function for portfolio selection is as follows (Kelly [1956]):

$$U = \ln(1 + r) \tag{1}$$

where

r = portfolio return,
ln = natural log function,

and the expected utility criterion leads to the following one-period optimization problem:

$$\text{Maximize } E(U) = \sum_{s} p_s U_s \tag{2}$$

where

s = scenario s, and the summation is over all scenarios,
p_s = probability of scenario s, and

U_s = utility in scenario s, based on the portfolio return in the scenario r_s, where the utility function is the log function in Equation (1).

The portfolio return in scenario s is calculated as a weighted average of the returns to the assets in the portfolio

$$r_s = \sum_i w_i r_{is} \tag{3}$$

where

i = asset i, and the summation is over all assets in the portfolio,
w_i = investment weight of asset i in the portfolio, and
r_{is} = return to asset i in scenario s.

The log utility function belongs to the family of power utility functions, which are defined as

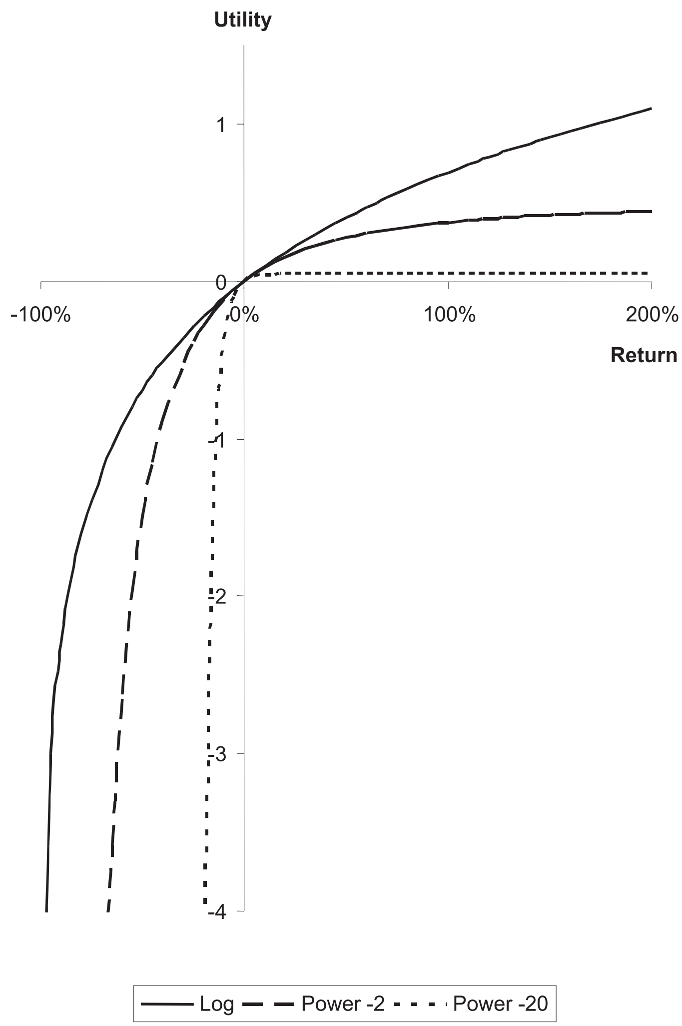
$$U = \frac{1}{\gamma}(1+r)^{\gamma} \tag{4}$$

where

r = portfolio return, and
γ = power, less than or equal to 1.

The log utility function is the special case of the power utility function with power gamma set to zero (Grauer and Hakansson [1982]). Figure 2 shows the log utility function and two other examples of power utility functions with powers of –2 and –20. These utility functions have a value of zero when the return is zero.[1] For gains (positive returns), the utility is positive; for losses (negative returns), it is negative. The power utility function with power 1 is risk-neutral. All power utility functions with powers less than 1 are risk-averse utility functions, including the log utility function.

FIGURE 2
Log and Power Utility Functions



The power utility function with a power of –2 shown in Figure 2 has greater risk aversion and greater curvature than the log utility function. Its penalty for losses also increases more quickly as losses increase. Finally, it provides greater downside protection, which is desirable for investors who have a greater aversion to losses than the aversion represented by the log utility function.

Unfortunately, the value that the power utility function with power –2 associates with gains is less than that of the log utility function. This decrease in the value of gains is an undesirable characteristic of the power utility functions with power less than zero.

The power utility function with a power of –20 shown in Figure 2 provides greater downside protection than the power utility function with power –2, but the value it associates with gains decreases even further relative to the log utility function. In general, as the power of the power utility function decreases,[2] downside protection increases relative to the log utility function. However, the value of gains also decreases relative to the log utility function.

The growth optimization with downside protection technique is based on a two-segment utility function, where the utility of gains is modeled with a log utility function and the utility of losses is modeled with a power utility function with power less than or equal to zero. This utility function combines the maximum growth characteristics of the log utility function on the upside with the scalable downside protection characteristics of the power function on the downside. We refer to it as the power-log utility function. It is defined as follows:

$$\begin{aligned} U &= \ln(1+r) \qquad \text{for } r \geq 0 \\ &= \frac{1}{\gamma}(1+r)^{\gamma} \qquad \text{for } r < 0 \end{aligned} \tag{5}$$

where

r = portfolio return, and
γ = power, less than or equal to 0.

Figure 3 gives an example of a power-log utility function with a downside power of –20. The utility function for gains is the log utility function; for losses it is the power function with a power of –20.

Investors can vary the level of downside protection they build into their portfolios by changing the downside power. Selecting a downside power of zero is equivalent to using a log utility function for losses. This will result in the construction of the maximum growth portfolio, since the utility function for gains is always a log utility function. Lower values of the downside power represent greater loss aversion, because the penalty for losses increases while the value associated with gains remains unchanged.

Across the entire range of returns, the power-log utility functions are characterized by utility that increases as returns increase, and marginal utility that decreases as returns increase. Gains and losses are valued with different utility functions when the downside power is less than zero. These utility functions allow more loss-averse investors to increase the penalty for losses, while leaving the utility of gains unchanged. As described by Kahneman and Tversky [1979], these power-log utility functions are "steeper in the negative than in the positive domain; losses loom larger than corresponding gains." Thus they conform to the Kahneman and Tversky postulates of reference dependence and loss aversion.

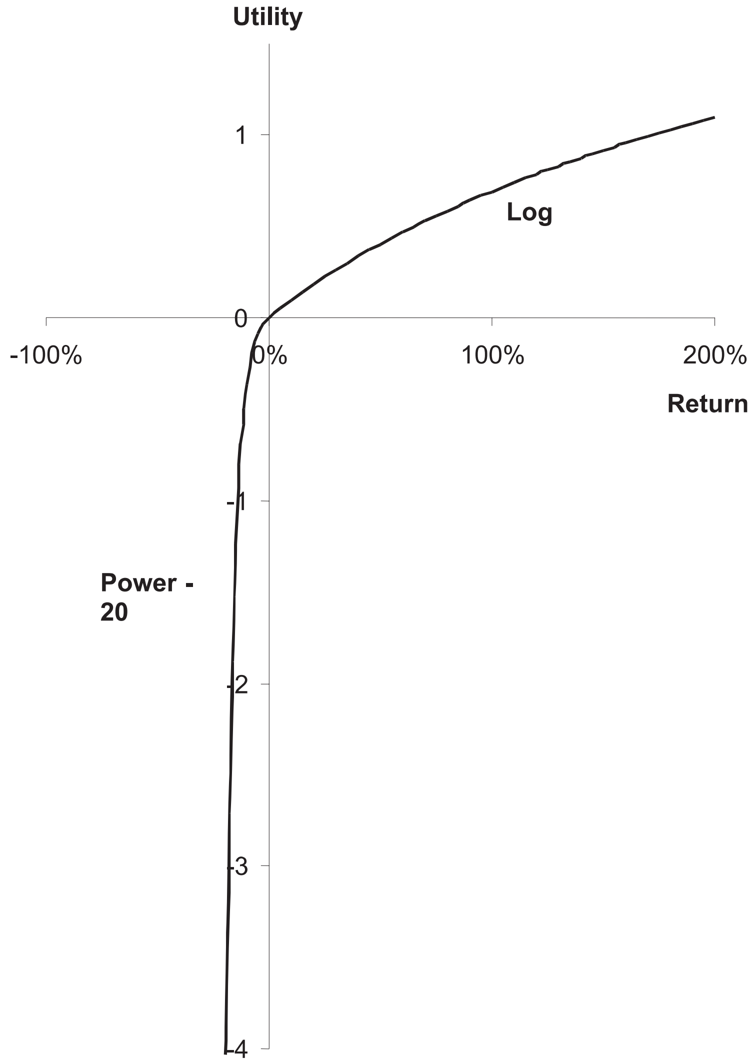
The power-log utility functions also conform to Kahneman and Tversky's postulate of diminishing sensitivity for gains. The marginal utility of the log utility function decreases as the size of the gain increases, represented by a concave function for gains. However, the power-log utility functions do not conform to the postulate of diminishing sensitivity for losses, which models risk-seeking behavior that would be represented by a convex function for losses. Rather, these utility functions are characterized by an increasing sensitivity to losses as the size of the losses increases, i.e., a concave function for losses, which represents risk-averse behavior.

Barberis and Thaler's [2002] survey provided evidence in support of risk-seeking behavior where losses are concerned. Their work was based on experiments and observation of speculative behavior. For examples, they used the one-shot gambles described in Kahneman and Tversky [1979], and the behavior of professional traders in the Treasury bond futures pit at the CBOT, where the time horizon of traders is one day (see Coval and Shumway [2000]).

In those experiments and observations, only a small fraction of the decision-maker's total wealth was at stake. Thus, a total loss in a one-shot gamble or trade would have little impact on the long-term well-being of the experimental subject or trader. Long-term investors, such as endowments or middle-aged investors making decisions about retirement funds, cannot afford to take a position that could result in a total loss. The power-log utility functions' increasing sensitivity to losses is a better representation of long-term investment behavior. In fact, the utility associated with a 100% loss is minus infinity.

For example, for the log utility function in Equation (1), which corresponds to a power utility function with power zero, when the return equals –100%, or –1, the utility value is log(0), which is minus infinity. The power-log utility functions will not allow selection of a portfolio where a 100% loss has a probability greater than zero. This is not true of prospect theory's S-shaped utility function, which is risk-seeking where losses are concerned (Figure 1).

FIGURE 3
Power-Log Utility Function with Downside Power –20



Another interesting characteristic of power-log utility functions is that they are continuously differentiable[3] across the entire range of returns. The slope of the log function used for gains is 1 when the return is zero. The slope of the power function used for losses is also 1 when the return is zero, for all powers less than or equal to zero. Thus, even though the reference-dependent, loss-averse power-log utility functions are steeper for negative than for positive returns, they do not have a kink at a return of zero. This allows development of fast optimization algorithms for portfolio selection.

The algorithm used here, a non-linear mathematical programming algorithm with a superlinear rate of convergence,[4] is based on an accelerated conjugate direction method developed by Best and Ritter [1976]. For a given power-log utility function, the optimal portfolio is selected by maximizing the expected utility in Equation (2). This requires the specification and use of the entire joint distribution of asset returns. As a result, all the moments of the distribution of asset returns, including mean, variance, skewness, and kurtosis, as well as all the correlations between returns, are taken into account implicitly. For example, if some assets have skewed return distributions, the optimization process produces portfolios with the positively skewed return distributions that reflect the growth maximization and loss aversion properties of the power-log utility functions.

To test the effectiveness of our portfolio selection method, we simulated asset returns for three assets with very different types of return distributions. The advantage of using simulated returns instead of historical data is that the results depend on the nature of the return distributions, rather than on the historical experience with particular types of assets and the associated time-period bias.

We started by assuming a return period of one year. For the first asset, we selected a riskless asset, and assumed a constant return of 4%. For the second asset, we chose a stock index that does not pay dividends. We assumed a beginning-of-year stock index price of $100, and that the distribution of the log of the stock index return is normal, with a 10% mean and a 20% standard deviation. To test the effectiveness of the process at creating portfolios with positively skewed return distributions, we also needed an asset with an asymmetric return distribution. Bonds, options, hedge funds, insurance policies, and bank loans were all suitable, as all have skewed return distributions and some have fat tails. We chose a call option as the third asset, since it is a familiar asset and has a skewed return distribution with a fat tail on the upside. We used an at-the-money European call option on the stock index, with one year to expiration, and calculated its beginning-of-year price of $9.9250 by using the Black-Scholes option pricing model.

To simulate the joint return distribution for the three assets, we generated 10,000 random returns from the stock index return distribution. Since that distribution is lognormal, the simulated return distribution is slightly skewed to the right, with a mean of 12.75%, a standard deviation of 22.77%, and skewness of 0.61.

To generate the call option returns that correspond to the stock index returns, we used the beginning-of-year stock index price of $100 and the simulated stock index returns to generate 10,000 corresponding end-of-year stock index prices. Next, we calculated the call option payoffs based on those 10,000 prices, and used the payoffs along with the beginning-of-year call option price of $9.9250 to calculate the corresponding call option returns. The call option's return distribution has a mean of 64.54%, a standard deviation of 188.48%, and skewness of 1.39. The correlation between the call option return and the stock return is 0.96.

We combined the return distributions for the riskless asset, the stock index, and the call option to create a joint distribution for the three assets with 10,000 observations. Each set of returns for an observation is assumed to be equally likely, with a 0.0001 probability of occurring. This joint return distribution is used for portfolio selection. Although the optimization algorithm we used here can create portfolios with long or short positions in any of the assets, we put in a "no-short sales" constraint on the stock index and the call option to make it easier to interpret the results.

Finally, to provide a perspective for the results obtained by using power-log utility functions, we also constructed optimal portfolios using classical mean-variance analysis and standard power utility functions. Note that mean-variance analysis may be considered inappropriate for constructing portfolios whose assets have return distributions with significant higher moments. However, it is the most widely used and familiar methodology for portfolio construction, and we feel it provides an interesting baseline for comparison. Comparing the optimal power-log portfolios with those produced using standard power utility functions is useful because power utility functions are commonly found in the literature.

The next sections compare the portfolio compositions and return characteristics of portfolios constructed using all three techniques.

Optimal Portfolios

Table 1 shows the portfolios constructed using power-log utility functions for downside powers of 0 to –50 in the optimal power-log portfolios panel; their expected returns are shown in the table's second column (Kale [2004]). The corresponding mean-variance (M-V) efficient portfolios and optimal power utility portfolios shown in the table have been constructed to match the expected return of the optimal power-log portfolios. We chose the expected portfolio return as the point of reference because it is an appropriate measure of reward in all cases. Other measures such as the standard deviation are not appropriate when comparing symmetric and asymmetric return distributions.

The optimal power-log portfolio in row 1 of Table 1 has been constructed with a downside power of 0. It is the maximum growth portfolio, since the power-log utility function with a downside power of 0 is the log utility function. It is a very risky, leveraged portfolio with a short position of 58.41% in the riskless asset, but is expected to have the highest growth rate over time. As the downside power is lowered, more downside protection is built into the portfolios, and the resulting optimal power-log portfolios are progressively more conservative. Remarkably, the weight of the call option relative to the weight of the stock in these portfolios increases as portfolio risk declines. This is due to the significant positive skewness in the call option's returns.

The M-V efficient portfolios shown in Table 1 have the lowest variance for the given expected returns. All have a negligible position in the call option. Clearly, the positive skewness of the call's returns carries no weight in constructing M-V efficient portfolios. The leveraged stock portfolios dominate the call option in mean-variance space. Even though the 0.96 correlation between the stock and call returns is less than 1, the resulting percent weight of the call option in all the M-V efficient portfolios is zero to two decimal places. The M-V efficient portfolio in row 1 uses a great deal of direct leverage, with a short position in the riskless asset of 104.97%, to achieve an expected return of 21.93%. For lower levels of expected return, the amount of leverage declines, until it is approximately zero in row 4. The corresponding power-log portfolio in Row 4 is very different: It has a large weight of 58.49% in the riskless asset, a smaller weight of 31.84% in the stock,

Table 1. *Optimal Portfolios*

		Optimal Power-Log Portfolios				M-V Efficient Portfolios			Optimal Power-Utility Portfolios			
Portfolio	Expected Return (%)	Downside Power	Riskless Weight (%)	Stock Weight (%)	Call Weight (%)	Riskless Weight (%)	Stock Weight (%)	Call Weight (%)	Riskless Weight (%)	Stock Weight (%)	Call Weight (%)	Power
1	21.93	0.00	–58.41	150.62	7.79	–104.97	204.97	0.00	–58.41	150.62	7.79	0.0000
2	17.85	–0.90	1.89	87.99	10.11	–58.36	158.36	0.00	–39.75	136.60	3.14	–0.2842
3	15.78	–1.70	27.46	62.11	10.43	–34.59	134.59	0.00	–26.54	125.18	1.36	–0.5045
4	12.66	–4.00	58.49	31.84	9.67	1.01	98.99	0.00	1.01	98.99	0.00	–1.0411
5	7.25	–50.00	93.31	1.56	5.13	62.85	37.15	0.00	62.85	37.15	0.00	–4.4197

and a significant weight of 9.67% in the call option. The return characteristics of these two portfolios, as well as the other pairs of optimal power-log and M-V efficient portfolios, are very different as well.

The compositions of the optimal power utility portfolios lie between the compositions of the corresponding power-log and M-V efficient portfolios. The optimal power utility portfolio in row 1 is the maximum growth portfolio, constructed by setting the power in the power utility function to zero, which makes it equivalent to a log utility function. It is identical to the power-log portfolio in row 1. The weight of the call option in the optimal power utility portfolios drops steadily as the power is decreased, going down to 0 in row 4, at which point it is virtually identical to the mean-variance efficient portfolio. These results suggest that for higher levels of risk aversion, the power utility method is similar to mean-variance analysis, while the power-log utility method is significantly different.

The last column of Table 1 shows the power used to construct the optimal power utility portfolio for the given expected return. Note that the power decreases more slowly than the downside power for the corresponding power-log portfolios. This is because a lower (more negative) power in the power utility function decreases the utility of upside returns and increases the penalty for downside returns (Figure 2). A lower downside power in the power-log utility function only increases the penalty for downside returns and leaves the utility of upside returns at their maximum growth value (Figure 3). As a result, a power utility function with a given power gives up more upside return than the power-log utility function with the same downside power, thus the expected return drops more quickly for optimal power utility portfolios as the power is lowered.

Table 2 compares the minimum, maximum, standard deviation, and negative semideviation below zero for the return distributions of the optimal portfolios. The composition and return distribution characteristics of the optimal power utility portfolios lie between those of the optimal power-log and the M-V efficient portfolios. We do not discuss these results for optimal power utility portfolios further, but they are shown in all the exhibits.

Table 2 shows that the entire range of returns for each optimal power-log portfolio lies above that for the corresponding M-V efficient portfolio. Thus, the power-log portfolios provide consistently better downside protection and greater upside potential than M-V efficient portfolios at each level of expected return.

Figure 4 shows the range of returns for the portfolios in Table 2 and for additional optimal portfolios at intermediate levels of expected return, for a total of thirty-four optimal portfolios of each type. Interestingly, for expected returns above 20%, all the M-V efficient portfolios have a minimum return of "–100%," i.e., bankruptcy. In other words, M-V efficiency can be fatal for high-risk, high-return portfolios! In contrast, the probability of bankruptcy for power-log portfolios is zero, based on the given joint distribution of asset returns. The utility of a –100% return is minus infinity, which means such a portfolio will not be constructed.

The standard deviation for the M-V efficient portfolios is slightly lower than the standard deviation for the corresponding power-log portfolios. This is because the M-V efficient portfolios have been constructed to minimize the standard deviation for a given expected return. However, standard deviation is an inappropriate measure of risk for assets with asymmetric return distributions. The power-log portfolios' larger positive deviations of return from their expected value, when compared to the M-V efficient portfolios, appear incorrectly as larger contributions to risk. These larger positive deviations are in fact highly desirable, and should not be penalized. For this reason, we do not use the standard deviation of return to compare the different portfolio construction techniques.

Negative semideviation is a better measure of risk than standard deviation. It focuses on downside returns only, and is a measure of downside risk (Markowitz [1959]). We define it as follows:

$$\textit{Negative Semideviation} = \sqrt{\sum_s p_s \left(r_s^-\right)^2} \tag{6}$$

where

r_s^- = the portfolio return in scenario s if it is negative, zero otherwise, and the sum is over all scenarios, and

p_s = the probability of scenario s.

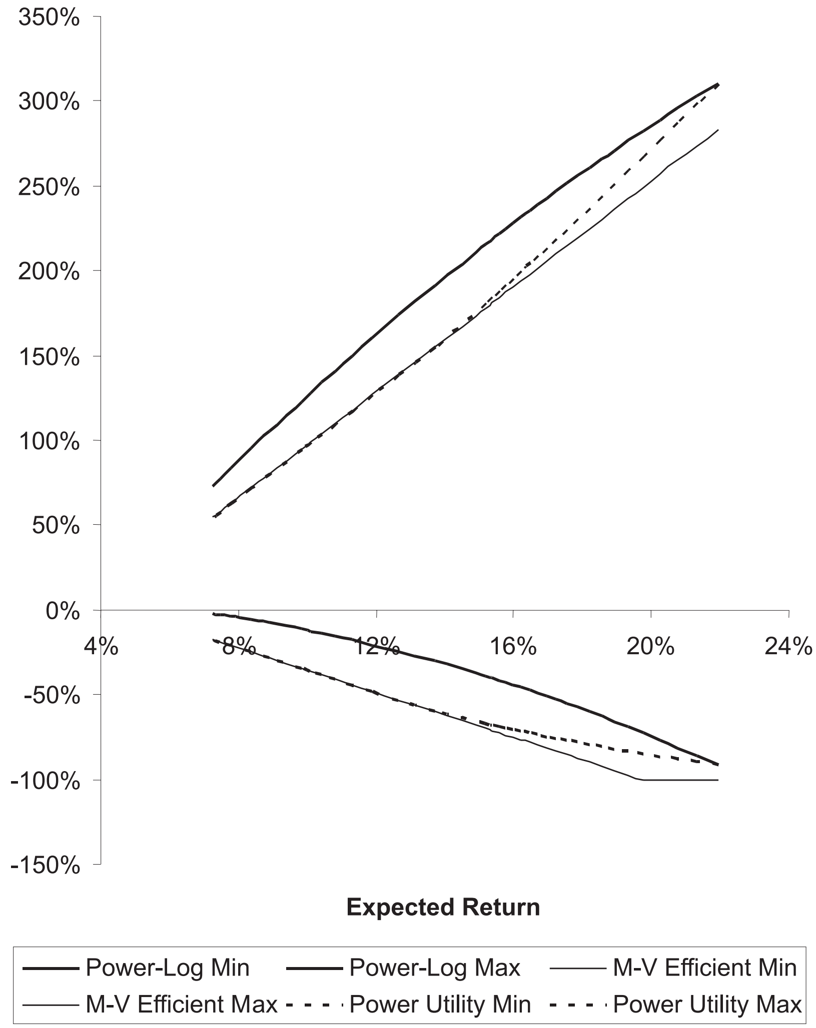
The last panel of Table 2 shows the negative semideviation for the optimal portfolios. It is smaller for all the power-log portfolios than for the corresponding M-V efficient portfolios. Based on this measure, the power-log portfolios have consistently lower risk than the M-V efficient portfolios at each level of expected return.

Table 3 compares the asymmetry characteristics of the return distributions of the optimal portfolios. Predictably, all the M-V efficient portfolios have exactly the same skewness as the stock return distribution, 0.61, since they are all leveraged stock portfolios with negligible call option positions. The skewness of the power-log portfolios increases consistently as risk decreases. The results clearly show the benefits of the power-log portfolios relative to the M-V efficient portfolios for investors who have a preference for positively skewed returns.

Table 2. *Range, Standard Deviation and Negative Semideviation of Returns for Optimal Portfolios*

		Minimum Return (%)			Maximum Return (%)			Standard Deviation (%)			Negative Semideviation (%)		
Portfolio	Expected Return (%)	Power-Log	M-V Efficient	Power-Utility	Power-Log	M-V Efficient	Power-Utility	Power-Log	M-V Efficient	Power-Utility	Power-Log	M-V Efficient	Power-Utility
1	21.93	–91.11	–100.00	–91.11	310.67	282.84	310.67	48.54	46.67	48.54	17.03	18.38	17.03
2	17.85	–57.35	–87.47	–78.18	255.89	219.42	230.92	38.69	36.06	36.82	12.15	13.78	13.19
3	15.78	–42.72	–73.75	–69.72	224.77	187.10	192.07	33.45	30.65	30.97	9.81	11.43	11.16
4	12.66	–24.45	–53.18	–53.18	173.67	138.66	138.66	25.25	22.54	22.54	6.46	7.93	7.93
5	7.25	–2.23	–17.46	–17.46	73.15	54.53	54.53	10.00	8.46	8.46	0.90	1.98	1.98

FIGURE 4
Range of Portfolio Returns



With the widespread use of derivatives and the need to control the risk exposure of portfolios and financial institutions, value at risk (VaR) has gained popularity as a measure of exposure to large losses (Smithson [1998]). VaR focuses on the probability and size of losses in the downside tail of the return distribution. At a 99% confidence level, if VaR is $5 million, there is a 1% probability of a loss of $5 million or higher.

Since the power-log portfolios have asymmetric return distributions, with fatter tails on the upside and thinner tails on the downside, a complementary measure for the potential value added on the upside is "value to gain," or VtG. At a 99% confidence level, if VtG is $50 million, there is a 1% probability of a gain of $50 million or higher. VaR and VtG are measures of location for the tails of the distribution, and can be used for dollar or percent losses and gains.

Table 3 shows the VaR and VtG for the optimal portfolios at the 99% confidence level. The VaR for the power-log portfolios is consistently lower at all levels of expected return than for the corresponding M-V efficient portfolios. The difference in VaR is especially dramatic for the most conservative portfolios. It appears the power-log portfolios do a better job of protecting investors against a large loss. The VtG for the power-log portfolios is also consistently higher than for the M-V efficient portfolios. In addition to better protecting the downside, the power-log portfolios also deliver higher potential gains.

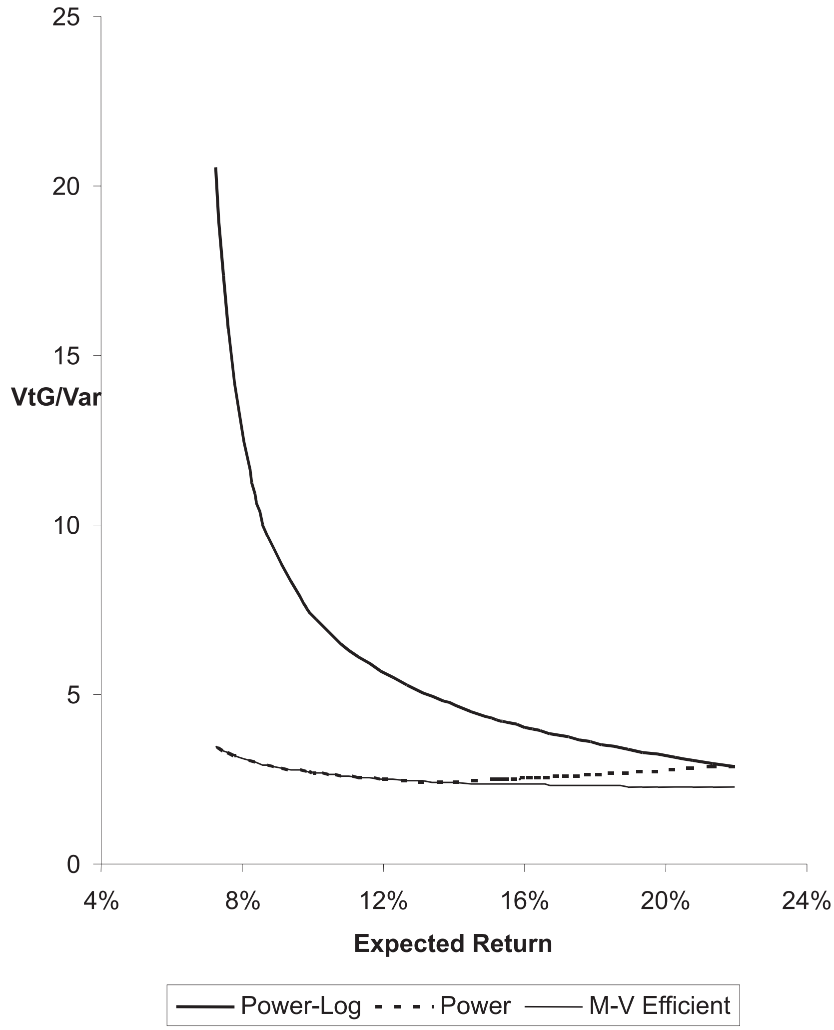
The ratio of VtG to VaR gives an idea of the asymmetry in the tails of the portfolio return distribution with regard to gains and losses. This is different from skewness, since skewness does not focus on the tails of the distribution exclusively.

The last panel of Table 3 and Figure 5 show the VtG to VaR ratios for the optimal portfolios at the 99% confidence level. At a given level of expected return, the ratio shows the size of the potential gains relative to the size of the potential losses at the extremes. As expected, the power-log portfolios dominate at all levels

Table 3. *Skewness, VaR, VtG and VtG/VaR Ratio of Returns for Optimal Portfolios*

		Skewness			Value at Risk (%)			Value to Gain (%)			Value to Gain/Value at Risk		
Portfolio	**Expected Return (%)**	**Power-Log**	**M-V Efficient**	**Power-Utility**	**Power-Log**	**M-V Efficient**	**Power-Utility**	**Power-Log**	**M-V Efficient**	**Power-Utility**	**Power-Log**	**M-V Efficient**	**Power-Utility**
1	21.93	0.90	0.61	0.90	55.71	66.24	55.71	161.08	148.99	161.08	2.89	2.25	2.89
2	17.85	1.06	0.61	0.77	36.67	50.26	46.08	131.88	116.01	121.03	3.60	2.31	2.63
3	15.78	1.13	0.61	0.69	28.13	42.12	40.31	115.61	99.20	101.37	4.11	2.36	2.51
4	12.66	1.23	0.61	0.61	16.97	29.92	29.92	89.27	74.02	74.02	5.26	2.47	2.47
5	7.25	1.37	0.61	0.61	1.87	8.73	8.73	38.39	30.28	30.28	20.55	3.47	3.47

FIGURE 5
VtG/Var Ratio for the Portfolios (99% Confidence Level)



of expected return, and dramatically more so for the conservative portfolios. As we move from risky to conservative portfolios, the M-V technique pulls in the tails of the distribution symmetrically, resulting in a relatively flat VtG/VaR ratio. The power-log technique pulls in the downside tail of the distribution much more than the upside tail, resulting in dramatically higher VtG/VaR ratios for the conservative portfolios.

Conclusion

This study combines the goal of long-term portfolio growth maximization with tenets of prospect theory such as reference dependence, loss aversion, and diminishing sensitivity to gains. We develop a portfolio selection technique based on the family of power-log utility functions. Our growth optimization with downside protection technique selects a portfolio by maximizing its expected utility for a given power-log utility function. Since it uses the entire joint distribution of asset returns, it accounts for the first, second, third, and higher moments as well as the comovement between returns. Thus, we can use it for all types of assets, including those whose return distributions are symmetric, skewed, fat-tailed, or any combination thereof. It will work with stocks, bonds, options, insurance assets, bank loans, structured securities, mutual funds, hedge funds, and with portfolios containing any or all of these assets. It can also be used to maximize portfolio growth for a wide range of investors, from those who are extremely risk-seeking to those who are extremely loss-averse.

To test the effectiveness of power-log utility functions for portfolio selection, we used three assets with very different types of return distributions: a riskless asset, a stock index, and a call option. We assumed the riskless asset had a constant return, the stock index had a lognormal return distribution, and the call option had a heavily skewed return distribution. We simulated the joint return distribution for the three assets, and constructed optimal portfolios ranging from a very risky maximum growth portfolio with a large leveraged position in the stock index and a big position in the call option, to a very conservative portfolio with a very large position in the riskless asset and a significant position in the call option. All the power-log portfolios had positively skewed return distributions, and their positive skewness increased progressively from the riskiest to the most conservative portfolios.

To put the results in perspective, we compared the optimal power-log portfolios to mean-variance efficient portfolios, with matched expected returns. At every level, the optimal power-log portfolios had consistently less downside risk and greater upside potential. The power-log portfolios also had higher positive skewness and much higher (value-to-gain/value-at-risk) ratios than the corresponding mean-variance efficient portfolios.

The growth optimization with downside protection technique using power-log utility functions gives investors a simple but powerful new way of constructing portfolios that reflect both desire for high growth over the long term and aversion to losses. It provides the flexibility of including any kind of asset in the portfolio, and produces portfolios with positively skewed returns if assets with skewed returns are included in the asset mix.

Future research will test the efficacy of this technique with data on traded assets, hedge funds, bank loan portfolios, and insurance portfolios. This technique should also be compared to other methods of portfolio selection besides mean-variance analysis and optimization with power utility functions. Particularly useful would be comparison to techniques that use measures of downside risk in optimal portfolio construction.

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Notes

1. The power utility functions have been transformed by subtracting a constant to fit this structure. The transformation does not change the utility associated with the power utility functions.
2. Power –20 is smaller than power –2.
3. Michael J. Best pointed out this characteristic of the power-log utility function.
4. This optimization algorithm was developed by Michael J. Best and the author.

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